1 Important Notations and foreknowledge

1.1 About the game

"The game alternates between Night and Day rounds, starting with the Night. At the start of Night, the Moderator tells all players, "Close your eyes." Everyone begins slapping their knees (or table) to cover up any noises of the Night. Then, the moderator prompts the following characters to wake up one by one...." – Stellar Factory (Click link for more detail rules)

1.2 Goal of this paper

Shentu, a master on this game, developed a scientific way to find the Werewolf (Similar for finding villagers and gods (including hunter, seer, doctor, etc.)). However, he only provided the proof by numerical simulation for his theorems; so, as a math Phd, I just want to give a formal proof of his theorems.

1.3 Important Notations

$\binom{n}{k} = C_n^k = \dfrac{n!}{k!(n-k)!}$, $k \leq n$	Number of ways to choose k elements from n elements
$\binom{n}{k_1, k_2 \cdots k_j} = \dfrac{n!}{k_1!k_2!\cdots k_j!}$ $k_i \leq n$, and $\sum_{i=1}^{j} k_i = n$	Number of ways to partition n elements into j stacks, with k_j elements in each stack
R_i	ith row of the matrix

2 Theorems

Theorem 2.1 (Three Row Theorem)**.** In general, there will be 12 people in the game, 4 of them are Gods, 4 of them are Werewolves, and the last 4 are normal villagers. If we regard 12 players as 12 entries in a 6×2 matrix, call it A, then

$$A = \begin{bmatrix} A_1 & A_7 \\ A_2 & A_8 \\ A_3 & A_9 \\ A_4 & A_{10} \\ A_5 & A_{11} \\ A_6 & A_{12} \end{bmatrix}$$

Now, we claim that: There is approximately 90% chance that the same identity (Gods, werewolves, or villagers) will appear in any three consecutive rows (including R_5, R_6, R_1 and R_6, R_1, R_2) of the matrix A.

Proof: The main idea of the proof is applying Inclusion - exclusion principle.

WLOG, we'll show it for Werewolves.

Firstly, there are 6 cases for the choice of 3 consecutive rows: R_1, R_2, R_3; R_2, R_3, R_4; R_3, R_4, R_5; R_4, R_5, R_6; R_5, R_6, R_1; R_6, R_1, R_2, and we named them case $C_1, C_2 \cdots C_6$ respectively. To make thing clear, we add two rows, R_7 and R_8, where $R_7 \equiv R_1$, and $R_8 \equiv R_2$, now we can deal with

$$A' = \begin{bmatrix} A_1 & A_7 \\ A_2 & A_8 \\ A_3 & A_9 \\ A_4 & A_{10} \\ A_5 & A_{11} \\ A_6 & A_{12} \\ A_1 & A_7 \\ A_2 & A_8 \end{bmatrix}$$

And, we calculate the total number of ways to put 4 werewolves into the matrix, which is $\binom{12}{4} = 495$

1. For any three rows, R_i, R_j, R_k, where $i \neq j \neq k$ the total number of combinations for at least three werewolves in these three rows can be divided into two cases

   - All four werewolves in these rows

     In this case, we must put 4 werewolves into this three rows (6 entries), so the total number of possible ways is $\binom{6}{4}$

   - Only three werewolves in these rows

     In this case, we must put 3 werewolves into this three rows (6 entries), and one in other entries except from these 6 ones, so the total number of possible ways is $\binom{6}{3}\binom{6}{1}$

   So for any three rows, R_i, R_j, R_k, where $i \neq j \neq k$ the total number of combinations for at least three werewolves in these three rows is $\binom{6}{4} + \binom{6}{3}\binom{6}{1}$

2. We can see that for $m > n$, where $m - n \equiv 1$, $C_m \cap C_n$ can be divided into three cases

   - At least three werewolves in row $\{R_n, R_{n+1}\}$

     Same logic as before, there should be $\binom{4}{4} + \binom{4}{3}\binom{8}{1}$ possible combinations.

   - Only two werewolves in row $\{R_n, R_{n+1}\}$, and one in row R_m, one in row R_{n+2} Same logic as before, there should be $\binom{4}{2}\binom{2}{1}\binom{2}{1}$ possible combinations.

   In conclusion, there are $\binom{4}{4} + \binom{4}{3}\binom{8}{1} + \binom{4}{2}\binom{2}{1}\binom{2}{1}$ ways.

3. Similarly, for $m > n$, where $m - n \equiv 1$, $C_m \cap C_n$, there are also two case, which is:

   - Two werewolves in row R_n, and for rest of two: one in row R_m and one in row R_{n+1}or one in row R_{m+1}, and one in row R_{n+2}

     Same logic as before, there should be $\binom{2}{2}\binom{2}{1}\binom{2}{1} \times 2$ possible combinations.

4. Finally, let's consider $m > n > k$, where $m - n = k - n \equiv 1$, $C_m \cap C_n \cap C_k$, there is only one case, which is

   - Two werewolves in row R_n, one in row R_m and one in row R_k

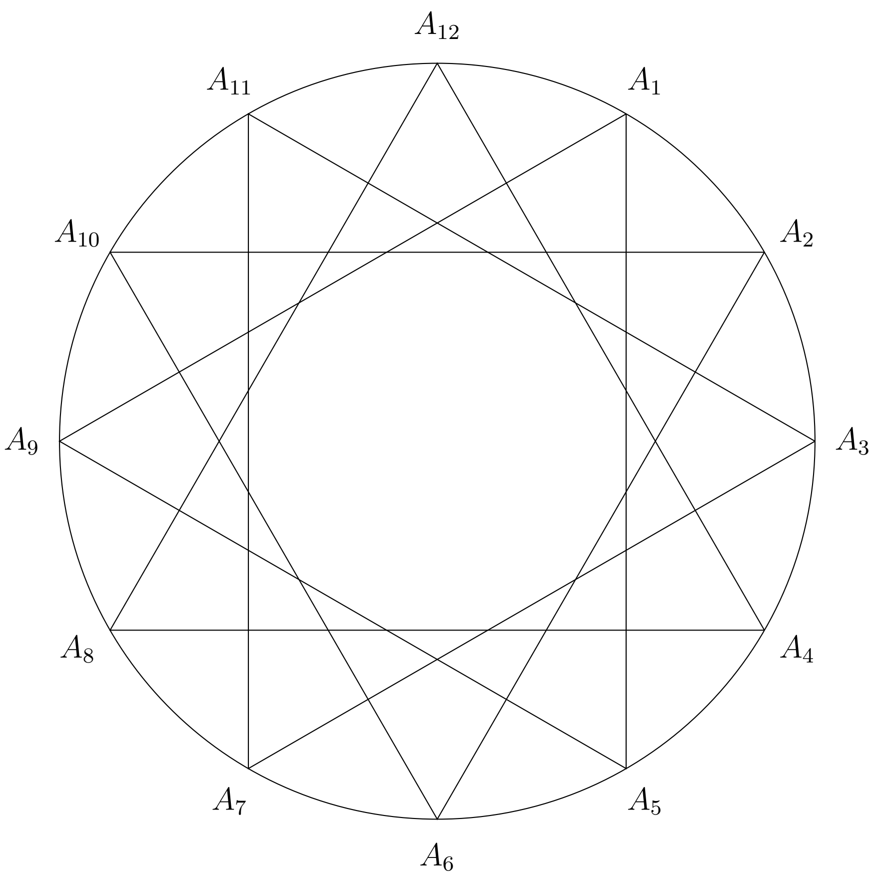
     Same logic as before, there should be $\binom{2}{2}\binom{2}{1}\binom{2}{1}$ possible combinations.

Now, we can use the inclusion-exclusion formula to calculate the total number of possible combinations for any three werewolves appearing 3 consecutive rows:

$$\begin{aligned}\left|\bigcup_{i=1}^{6} C_i\right| &= \sum_{i=1}^{6}|C_i| - \sum_{i<j}|C_i \cap C_j| + \sum_{i<j<k}|C_i \cap C_j \cap C_k| - \cdots - |C_1 \cap C_2 \cap \cdots C_6| \\ &= 6 \times \left(\binom{6}{4} + \binom{6}{3}\binom{6}{1}\right) - 6 \times \left(\binom{4}{4} + \binom{4}{3}\binom{8}{1} + \binom{4}{2}\binom{2}{1}\binom{2}{1} + \binom{2}{2}\binom{2}{1}\binom{2}{1} \times 2\right) + 6 \times \binom{2}{2}\binom{2}{1}\binom{2}{1} \\ &- 0 + \cdots - 0 \\ &= 6 \times \left(\binom{6}{4} + \binom{6}{3}\binom{6}{1}\right) - 6 \times \left(\binom{4}{4} + \binom{4}{3}\binom{8}{1} + \binom{4}{2}\binom{2}{1}\binom{2}{1} + \binom{2}{2}\binom{2}{1}\binom{2}{1} \times 2\right) + 6 \times \binom{2}{2}\binom{2}{1}\binom{2}{1} \\ &= 444\end{aligned}$$

In conclusion, the desired possibility is $\frac{444}{495} \approx 90\%$

Theorem 2.2 (Triangle Theorem)**.** If 12 players sit in a circle, and label them by $A_1 \cdots A_{12}$, namely



We can see that there are four triangle inside the circle, and 4 werewolves are more likely to sit in the pattern of 2 of them in the same triangle and the other two in two different triangles.
Proof: **Omitted**, similar to the first one, and simpler.